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S.M. Nikol'skii's Works on the Theory of Function Spaces and Its Applications[1]

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In 1951, S.M. Nikol'skii published his fundamental work [1], which laid a basis for a whole direction in the study of the theory of spaces of differentiable functions and its applications. In this paper, the following inequalities were obtained for the entire functions of finite order

$$g_\nu(z) = g_\nu(z_1, \dots, z_n), \qquad \nu = (\nu_1, \dots, \nu_n):$$

$$\|g_\nu\|_{q,\mathbb{R}^n} \le 2^n \left(\prod_{j=1}^{n} \nu_j\right)^{\frac{1}{p}-\frac{1}{q}} \|g_\nu\|_{p,\mathbb{R}^n}, \qquad 1 \le p \le q \le \infty, \tag{1}$$

$$\|g_\nu\|_{p,\mathbb{R}^m} \le 2^n \left(\sum_{j=m+1}^{n} \nu_j\right)^{\frac{1}{p}} \|g_\nu\|_{p,\mathbb{R}^n}, \qquad 1 \le m < n, \tag{2}$$

where $\|\cdot\|_{p,\mathbb{R}^n}$, $\|\cdot\|_{p,\mathbb{R}^m}$ are the Lebesgue norms in $\mathbb{R}^n$ and in its subspace $\mathbb{R}^m = \{x_1, \dots, x_m, 0, \dots, 0\}$, respectively.

These inequalities, combined with the estimates for derivatives

$$\|D^\alpha g_\nu\|_{p,\mathbb{R}^n} \le \nu^\alpha \|g_\nu\|_{p,\mathbb{R}^n}, \qquad 1 \le p < \infty, \tag{3}$$

that are proved by Nikol'skii and generalize Bernstein's inequality for $p = \infty$, allowed Sergei Mikhailovich to relate the approximation properties of functions from $L_p(\mathbb{R}^n)$ to their smoothness in $L_p(\mathbb{R}^n)$ (including the smoothness of fractional order) and apply the results obtained to the analysis of the spaces H_p^r, of differentiable functions, introduced by him. These spaces are normed as follows:

$$\|f\|_{H_p^r(\mathbb{R}^n)} = \|f\|_{p,\mathbb{R}^n} + \sup_{|h|>0} \frac{\|\triangle_h^k \nabla^{\overline{r}} f\|_{p,\mathbb{R}^n}}{|h|^{r-\overline{r}}}, \tag{4}$$

where $r > 0$, $\overline{r} \in \mathbb{N}_0$, $0 \le \overline{r} < r \le \overline{r} + 1$, $k \in \{1, 2\}$, $k > r - \overline{r}$, $\nabla^m f$ is the mth gradient of f, and $\triangle_h^k f(x)$ is the kth difference of f at the point x with step $h \in \mathbb{R}^n$.

Sergei Mikhailovich proved the following embedding theorem:

$$H_p^r(\mathbb{R}^n) \subset H_q^\rho(\mathbb{R}^m), \qquad 1 \le p \le q \le \infty, \quad 1 \le m \le n, \quad r - \frac{n}{p} = \rho - \frac{m}{q}. \tag{5}$$

This theorem is exact (unimprovable with respect to any parameter).

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The case $1 \le m < n$, $q = p$ shows that the trace of a function from $H_p^r(\mathbb{R}^n)$ on $\mathbb{R}^m$ belongs to $H_p^\rho(\mathbb{R}^m)$, $\rho = r - \frac{n-m}{p} > 0$. It is remarkable that this assertion is invertible, as it was demonstrated by Nikol'skii in 1951 [1] while solving a more general problem of constructing a function $f \in H_p^r(\mathbb{R}^n)$ with prescribed traces on $\mathbb{R}^m$ of the function itself and all its normal derivatives up to a certain order. Thus, an exact (invertible) characteristic of the trace of a function on a manifold of a lesser number of dimensions was obtained for the first time:

$$H_p^r(\mathbb{R}^n)|_{\mathbb{R}^m} = H_p^\rho(\mathbb{R}^m), \qquad \rho = r - \frac{n-m}{p} > 0. \tag{6}$$

The embedding theorems for the spaces of functions of several real variables were proved for the first time by S.L. Sobolev in the end of 1930s as applied to the function spaces

$$W_p^l(\mathbb{R}^n), \quad l \in \mathbb{N}, \qquad \|f|W_p^l(\mathbb{R}^n)\| = \sum_{|\alpha| \le l} \|D^\alpha f\|_{p,\mathbb{R}^n},$$

introduced by him. These theorems, including a supplement of V.P. Il'in, are expressed as

$$\begin{gathered} W_p^l(\mathbb{R}^n) \subset L_q(\mathbb{R}^m), \qquad 1 < p \le q \le \infty, \\ l - \frac{n}{p} + \frac{m}{q} \ge 0 \quad \text{for} \ \ q < \infty, \qquad l - \frac{n}{p} > 0 \quad \text{for} \ \ q = \infty. \end{gathered} \tag{7}$$

Embedding (7) is unimprovable in terms of the Sobolev and Lebesgue spaces involved in its formulation. However, there remained unsolved certain important problems in the embedding theory of Sobolev spaces. For example, one cannot obtain the inverse of the theorem on traces (which would enable one to characterize in an internal way the traces of functions from Sobolev spaces on a subspace of $\mathbb{R}^n$).

The studies of Sergei Mikhailovich marked an important stage in the development of the embedding theory of spaces of differentiable functions that determined a future trend for tens of years. Briefly, the importance of the theory of function spaces developed by Sergei Mikhailovich consists in the following.

1. This theory widely employs the spaces not only of integer but also of fractional smoothness; this detail in the characterization of functions is important for applications.

2. For integer l, the spaces $H_p^l(\mathbb{R}^n)$ constructed are arbitrarily close to the Sobolev spaces:

$$H_p^{l+\varepsilon}(\mathbb{R}^n) \subset W_p^l(\mathbb{R}^n) \subset H_p^l(\mathbb{R}^n) \qquad \forall \varepsilon > 0.$$

3. The inverse of the theorem on traces was obtained for the first time.

4. The system $\{H_p^r(\mathbb{R}^n)\}$ is closed with respect to the embedding theorems and the inverse theorems on the extension of functions.

5. The embedding theory includes anisotropic spaces of differentiable functions

$$H_{\mathbf{p}}^{\mathbf{r}}(\mathbb{R}^n), \qquad \mathbf{r} = (r_1, \dots, r_n), \quad r_i > 0, \quad \mathbf{p} = (p_1, \dots, p_n), \quad p_i \in [1, \infty].$$

To describe the results of Nikol'skii on anisotropic spaces of differentiable functions of several variables, we introduce certain notation. Denote by

$$\triangle_{i,h}^k f(x), \qquad i \in \{1, \dots, n\}, \quad h \in \mathbb{R}, \quad k \in \mathbb{N},$$

the difference, with step h of order k, of the function f at point x with respect to variable x_i. Let $\mathbf{p} = (p_1, \dots, p_n) \in [1, \infty]^n$ and $\mathbf{r} = (r_1, \dots, r_n) \in (0, \infty)^n$.

The space $H_{\mathbf{p}}^{\mathbf{r}}(\mathbb{R}^n)$ is a Banach space of functions f defined on $\mathbb{R}^n$ with the finite norm

$$\|f|H_{\mathbf{p}}^{\mathbf{r}}(\mathbb{R}^n)\| = \sum_{i=1}^{n}\left(\|f|L_{p_i}(\mathbb{R}^n)\| + \sup_{|h|>0}|h|^{\overline{r}_i - r_i}\left\|\triangle_{i,h}^{k_i}\frac{\partial^{\overline{r}_i}}{\partial x_i^{\overline{r}_i}}f \;\Big|\; L_{p_i}(\mathbb{R}^n)\right\|\right), \tag{8}$$

where $\overline{r}_i \in \mathbb{N}_0$, $0 \le \overline{r}_i < r_i \le \overline{r}_i + 1$, $k_i \in \{1, 2\}$, and $k_i > r_i - \overline{r}_i$.

When all p_i are identical ($p_1 = \ldots = p_n = p$), one writes $H_p^{\mathbf{r}}(\mathbb{R}^n)$ instead of $H_{\mathbf{p}}^{\mathbf{r}}(\mathbb{R}^n)$; if, in addition, $r_1 = \ldots = r_n = r$, then one writes $H_p^r(\mathbb{R}^n)$. In the latter case, norm (8) contains only the unmixed derivatives of the function f and the differences taken along the coordinate axes. However, norm (8) in this case is equivalent to norm (4) by virtue of the theorem proved by Nikol'skii on the properties of the mixed derivatives of functions from the space $H_{\mathbf{p}}^{\mathbf{r}}(\mathbb{R}^n)$. For $p_1 = \ldots = p_n = p$, this theorem states that any function $f \in H_p^{\mathbf{r}}(\mathbb{R}^n)$ has a derivative $D^\alpha f$, $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}_0^n$, on $\mathbb{R}^n$, provided that

$$\sum_{i=1}^{n}\frac{\alpha_i}{r_i} < 1;$$

moreover, the following estimate is valid:

$$\|D^\alpha f|H_p^{\boldsymbol{\rho}}(\mathbb{R}^n)\| \le C\|f|H_p^{\mathbf{r}}(\mathbb{R}^n)\|, \tag{9}$$

where

$$\boldsymbol{\rho} = (\rho_1, \ldots, \rho_n) \in (0, \infty)^n, \qquad \rho_j = r_j\left(1 - \sum_{i=1}^{n}\frac{\alpha_i}{r_i}\right),$$

and the constant C is independent of f.

Estimate (9) was also proved for functions from $H_{\mathbf{p}}^{\mathbf{r}}(\mathbb{R}^n)$; it was the first general interpolation theorem for differentiable functions of several variables when the "intermediate" properties of a function are induced by its other differential properties of different orders that are described in different L_p-metrics.

Nikol'skii established a general theorem on the embedding $H_{\mathbf{p}}^{\mathbf{r}}(\mathbb{R}^n) \subset H_{\mathbf{q}}^{\boldsymbol{\rho}}(\mathbb{R}^m)$. When $p_1 = \ldots = p_n = p$ and $q_1 = \ldots = q_m = q$, this theorem states that, for

$$1 \le m \le n, \quad 1 \le p \le q \le \infty, \quad \kappa = 1 - \left(\frac{1}{p} - \frac{1}{q}\right)\sum_{i=1}^{n}\frac{1}{r_i} - \frac{1}{p}\sum_{i=m+1}^{n}\frac{1}{r_i} > 0,$$

the embedding

$$H_p^{\mathbf{r}}(\mathbb{R}^n) \subset H_q^{\kappa\mathbf{r}^{(m)}}(\mathbb{R}^m) \tag{10}$$

holds, where $\mathbf{r}^{(m)} = (r_1, \ldots, r_m)$ is the projection of $\mathbf{r}$ onto $\mathbb{R}^m$.

He also obtained the inverse of theorem (10) for the case $1 \le m < n$ and $p = q$, which yields an exact (invertible) characteristic of the trace of functions $f \in H_p^{\mathbf{r}}(\mathbb{R}^n)$ on $\mathbb{R}^m$ (the extension of relation (6) to the anisotropic case).

Sergei Mikhailovich found the conditions of the compactness of sets in the space $H_p^{\mathbf{r}}(\mathbb{R}^n)$.

He extended his results to the spaces $H_p^r(G)$ defined on a domain $G \subset \mathbb{R}^n$ with a sufficiently smooth boundary ∂G [2]. In this case, the norm in $H_p^r(G)$ is given by

$$\|f\|_{H_p^r(G)} = \|f\|_{p,G} + \sup_{h\in\mathbb{R}^n,\ 0<|h|<\delta}\frac{\|\triangle_h^k\nabla^{\overline{r}}f\|_{p,G_{k\delta}}}{|h|^{r-\overline{r}}};$$

here, as in (4),

$$r > 0, \quad \overline{r} \in \mathbb{N}_0, \quad 0 \le \overline{r} < r \le \overline{r} + 1, \quad k \in \{1, 2\}, \quad k > r - \overline{r},$$

$\delta > 0$ is sufficiently small, and $G_\delta = \{x\colon \operatorname{dist}(x, \mathbb{R}^n \backslash G) > \delta\}$.

Nikol'skii constructed a continuous linear operator of extension

$$H_p^r(G) \to H_p^r(\mathbb{R}^n), \tag{11}$$

which allowed him to apply the theory of spaces $H_p^r(\mathbb{R}^n)$ to spaces $H_p^r(G)$.

Sergei Mikhailovich found an exact (invertible) characteristic of the traces of functions from $H_p^r(G)$ on a sufficiently smooth boundary ∂G [2]:

$$H_p^r(G)|_{\partial G} = H_p^\rho(\partial G). \tag{12}$$

He also found an exact (invertible) characteristic of the traces of functions from $H_p^r(G)$ on the boundary of a plane domain G with corner points.

The results obtained by him on the theory of function spaces are contained in the monographs [3, 4].

Sergei Mikhailovich applied his results on function spaces to equations of mathematical physics and, first of all, to the variational method of solving problems.

Nikol'skii was the first to establish the sufficient solvability conditions, in terms of boundary data, that differ in smoothness from the necessary conditions "by an arbitrarily small $\varepsilon > 0$." For example, for the boundary value problem for the Laplace equation

$$\begin{cases} \Delta u = 0 \quad \text{on } G, \\ u|_{\partial G} = \phi, \end{cases}$$

where the domain $G \subset \mathbb{R}^n$ has a sufficiently smooth boundary ∂G, to be solvable by a variational method, it is necessary that $\phi \in H_2^{\frac{1}{2}}(\partial G)$ and sufficient that $\phi \in H_2^{\frac{1}{2}+\varepsilon}(\partial G)$.

A separate series of works of Sergei Mikhailovich (1957–1959) was devoted to the solution of the Hilbert problem by the variational method (a problem of finding a function subject to certain conditions that is harmonic outside a bounded domain $G \subset \mathbb{R}^n$). The application of the variational method allowed him to analyze the above problem in the multidimensional case ($n \ge 2$), whereas the earlier used theory of the Cauchy integral is effective only for $n = 2$.

Sergei Mikhailovich studied the spaces of functions with a dominating mixed derivative $S_p^r H$. In these spaces, the properties of certain mixed derivatives play a decisive role. He established embedding theorems and other properties for these spaces and applied these studies to the proof of the existence and to the study of the properties of the solution to the first boundary value problem for the equation

$$\sum_{k,l \in E} (-1)^{|l|} (a_{kl}(x) u^{(k)}(x))^{(l)} = 0,$$

where the set E of multiindices is convex and contains all projections of each of its multiindices. Here, it was assumed that the condition

$$\int_G \sum_{k,l \in E} a_{kl}(x) f^{(k)}(x) f^{(l)}(x) dx \ge \kappa \int_G \sum_{k=1}^N (f^{(k^s)}(x))^2 dx$$

holds, where the convex envelope of the set $\{0\} \cup \{k^s\}_1^N$ coincides with E. The class of such equations falls outside the class of hypoelliptic equations.

The weighted spaces and their applications to degenerate elliptic equations were the subject of the series of works of S.M. Nikol'skii and P.I. Lizorkin started in 1964. In this works, an important role is played by the inequality

$$\|f\|_{L_p(G)} \le C \left(\sum_{j=0}^{s-1} \left\| \frac{\partial^j f}{\partial \nu^j} \right\|_{L_p(\partial G)} + \sum_{|k|=r} \|\rho^\lambda f^{(k)}\|_{L_p(G)} \right).$$

Here, the domain $G \subset \mathbb{R}^n$, $\rho(x) = \operatorname{dist}(x, \partial G)$, ν is a normal on ∂G, $\frac{r}{2} \le s \le r$, and $r - \frac{1}{p} - \lambda < s < r - \frac{1}{p} - \lambda + 1$.

After proving this inequality for a straight-line segment, Sergei Mikhailovich proposed the so-called method of bridges that allows one to prove the above inequality for a domain.

Nikol'skii and Lizorkin applied these results to strongly degenerate elliptic equations of order $2r$:

$$\begin{cases} Lu = F \quad \text{on } G, \\ \left.\dfrac{\partial u^j}{\partial \nu^j}\right|_{\partial G} = \phi_j, \quad j = 0, 1, \dots, s-1, \end{cases}$$

where

$$Lu = \sum_{|k|,|l| \le r} (-1)^{|k|} (a_{kl}(x) u^{(l)}(x))^{(k)}, \qquad a_{kl} = a_{lk},$$

$$\sum_{|k|,|l| \le r} a_{kl}(x) \xi_k \xi_l \ge \frac{\kappa}{\rho(x)^{2\lambda}} \sum_{|k|=r} \xi_k^2 \qquad \forall \xi_k.$$

Under strong degeneration ($\lambda < -\frac{1}{2} \;\Rightarrow\; s < r$), the number of boundary conditions is less than r on the whole boundary; this fact distinguishes this statement of the problem from the classical one.

The series of works by Sergei Mikhailovich carried out during 1991–1999 was devoted to the approximations of functions on manifolds by trigonometric polynomials, entire functions, and algebraic polynomials. The general approximation theorems were established for the manifolds $\Gamma = \Gamma^{(m)} \subset \mathbb{R}^n$, $1 \le m \le n$, for which there are theorems on the extension of functions from the space $H_p^r(\Gamma)$ to functions from the space $H_p^\rho(\mathbb{R}^n)$, $\rho = r + \frac{n-m}{p}$.

Note, for example, a theorem on the approximation of a function by algebraic polynomials:

$$f \in H_p^r(\Gamma)$$

is equivalent to the existence of a sequence $\{P_N\}_1^\infty$ of polynomials such that

$$\|f - P_N\|_{L_p(\Gamma)} \le C N^{-r}, \tag{$*$}$$

$$\|P_N\|_{H_p^r(\Gamma)} \le C. \tag{$**$}$$

This theorem contains a new statement of the problem and a complete result. What is new here as compared with the classical statement (Jackson, Bernstein, and de la Valleé Poussin) is condition ($**$). Note that, in the classical statement of the problem, when condition ($**$) is replaced by the condition

$$\|P_N\|_{L_p(\Gamma)} \le C, \tag{$***$}$$

the theorem is not valid any longer even for the interval $\Gamma = (0,1) \subset \mathbb{R}^1$, when conditions ($*$) and ($***$) imply $f \in H_p^{\frac{r}{2}}(\Gamma)$ but not $f \in H_p^{\frac{r}{2}+\varepsilon}(\Gamma)$ for any $\varepsilon > 0$.

Along with these results, Sergei Mikhailovich described manifolds Γ such that conditions ($*$) and ($***$) imply $f \in H_p^r(\Gamma)$.

In 1999, Sergei Mikhailovich published a work devoted to the following boundary value problem for polynomials for a self-adjoint elliptic differential equation of order $2l$:

$$\begin{cases} Lu = 0 \quad \text{on } G \subset \mathbb{R}^n, \\ \dfrac{\partial^m u}{\partial \nu^m}\Big|_{\partial G} = \dfrac{\partial^m P}{\partial \nu^m}\Big|_{\partial G}, \quad m = 0, 1, \dots, l-1, \end{cases}$$

where P is a polynomial.

It is shown that, if ∂G is a sphere or an ellipsoid, then the solution is a polynomial of the same degree as P, whereas, if ∂G is an algebraic surface of order $2s > 2$, then the solution may not be a polynomial. The results obtained in this work can be considered as a generalization of the main theorem for spherical functions (in the latter, $L = \Delta$, $l = 1$, and ∂G is a sphere).

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